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LETTER TO THE EDITOR

Dynamical properties of the two- and three-dimensional Ising models by 'damage spreading'

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Abstract. The method of 'damage spreading' is used to measure the dynamic exponent z for both the two- and three-dimensional Ising models with heat bath dynamics. We also measure a new exponent α describing the rate at which a damage cloud grows. We report values of $z = 2.24 \pm 0.04$ and $\alpha = 0.77 \pm 0.11$ in two dimensions, and $z = 2.202 \pm 0.03$ and $\alpha = 0.92 \pm 0.10$ in three dimensions.

'Damage spreading' is a useful method for determining the onset of chaotic behaviour in the Kauffman cellular automata and related problems [1, 2]. This method involves the study of the time development of the differences, or 'damage', between two systems which are nearly identical at some time $t = t_0$, and which are subsequently subjected to the same set of dynamics and evolutionary constraints. Since both systems are subjected to the same set of constraints and dynamics then the damage, if any, which develops between them as they evolve, is due to the initial perturbation by which the two systems were not *completely* identical at $t = t_0$. This form of analysis may be considered as a careful experiment, albeit a computer experiment, in which there is a 'control' system and a 'subject' system which only differ due to the introduction of some perturbation. Since this a computer experiment, it is possible to ensure that all events in the evolution of both systems before and after the perturbation is introduced, are identical. For example, in an Ising model, this would mean using the same dynamics, the same sequence of random numbers, and updating sites in the same order, in two systems which are initially different by only the value of one spin. In such a case, it is possible in the computer to isolate and trace the full influence of an initial perturbation on the evolution of the system.

Consider damage spreading as applied to the Kauffman model. The set of rules that govern the evolution of each site is selected with a preset probability p, from the 2^{16} rules that are possible for each site on the square lattice with nearest-neighbour interactions. A clone is made of this system with identical rules on all the sites, apart from a special site j where the rules are different between the two systems. It is possible to study how these two nearly identical systems evolve away from the same initial spin configuration. The standard questions considered are as follows. (i) Will the initial damage create an effect at the extremities of the system, or will it be localised? That is, is the system chaotic in the sense that two nearby points in the systems phase space

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will remain close together or move in independent trajectories? (ii) What is the rate of spread of damage? (iii) How many sites are infected on average as a function of time?

The method of damage spreading has also been applied to equilibrium thermodynamic systems [3, 4]. Coniglio *et al* [5] showed that if an initial perturbation is defined appropriately in a ferromagnetic Ising model, then the subsequent damage could be analysed to give well known static equilibrium properties. As an example consider two identical copies of an Ising model, labelled A and B, in dimension $d=2$ at a temperature T. Damage is introduced at the boundaries of the system by setting the boundary spins of system A permanently up, while keeping those of system B permanently down. Initially all other spins in both systems are in identical states. The systems evolve by heat bath dynamics, and equivalent sites of both systems are visited at the same time, with the same random number being used to determine the new state of each spin. Hence the interaction with the heat bath is identical for both systems. An interior site k of the system is defined to be damaged if the value of the spin at k in system A is different from the spin value at k in system B. In a nearest-neighbour model such as this, any damaged site must be connected to the initial damage by a continuous path of sites which have been damaged at least once in the time since the initial damage was introduced. The average damage between the two systems at equilibrium is a measure of the order parameter, which is non-zero below T_c and vanishes at T_c. Above T_c the damage is localised at the border and is zero in the interior of the lattice, which reproduces the behaviour of the order parameter expected from the exact Ising solution for an infinite-sized system. This effect is difficult to observe with standard Monte Carlo tehcniques [6]. Damage spreading is a very sensitive method for detecting differences that evolve between the systems due to slightly different initial conditions.

In this letter we show how damage spreading can be used to study dynamical properties. In particular we measure the dynamical exponent, z, for the $d=2$ and $d=3$ Ising models.

The standard Monte Carlo method for measuring z is through the time dependence autocorrelation function for the magnetisation, m, in equilibrium: it is expected that

$$\langle m(0)m(t)\rangle - \langle m\rangle^2 \sim \exp(-t/\tau) \tag{1}$$

where

$$\tau \sim \xi^z \tag{2}$$

and t is the time, ξ is the correlation length, and τ is the relaxation time of the slowest mode of relaxation of the system in equilibrium. A possible problem with this approach is a clean and precise determination of $\langle m(0)m(t)\rangle$ for large t. Figure 1 shows the typical behaviour of the autocorrelation function with time, as measured in units of Monte Carlo steps. Three regions are clearly observed; a relatively sharp decay, a second region where the decay is less sharp but well defined, and a third region where there is still some apparent decay but which is masked by oscillatory behaviour. Region II is usually used for extracting τ, and a careful study [7] uses, for example, a two-parameter fit which assumes that the slowest and the next slowest modes are responsible for the decay of the time-dependent correlation function in region II. Monte Carlo methods and various renormalisation group techniques support the values of $z=2.14\pm0.05$ ($d=2$) [8] and $z=2.02\pm0.03$ ($d=3$).

In this work we use damage spreading to measure the average time τ_e, taken for fluctuations to induce damage at the edge of an Ising system due to an initial central

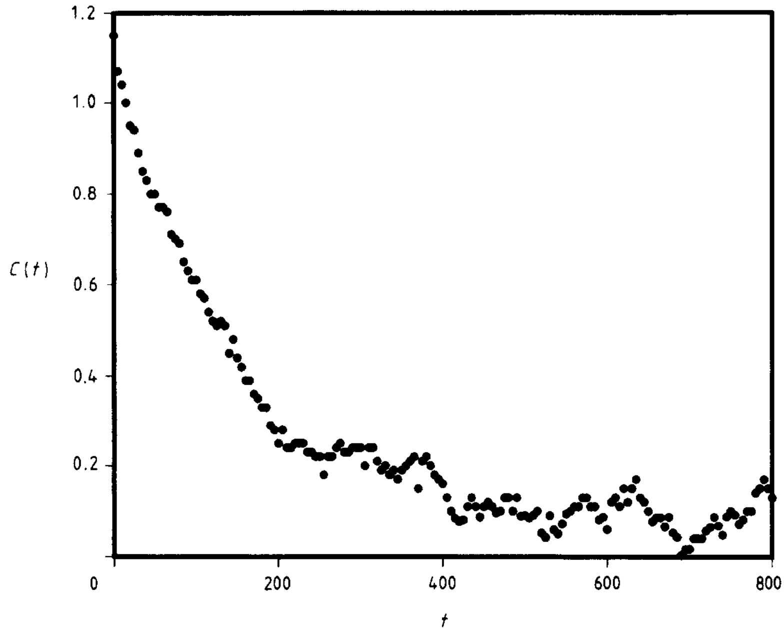

Figure 1. Plot of the magnetisation autocorrelation function

$$C(t) = (\langle m(0)m(t)\rangle - \langle m\rangle^2)/(\langle m^2\rangle - \langle m\rangle^2)$$

for a $d = 2$ Ising model of linear size $L = 31$. The time t is measured in Monte Carlo steps.

perturbation. Thus defined, τ_e is the average time taken for a signal to propagate from the interior to the edge of the lattice at equilibrium. Since the transmission of information across the entire system is dominated by the slowest mode of relaxation, we assert that τ_e is a measure of the characteristic time of the slowest mode of relaxation, and that τ_e therefore scales with ξ in the same way as τ in (2). For $t \ll \tau_e$ the system is highly correlated to the initial equilibrium state, while for $t \gg \tau_e$ the system becomes decorrelated. We find that in $d = 2$, τ_e as measured above is of the same order as the value of τ found by the conventional method, but in $d = 3$ we find that the τ_e values are smaller by a factor of 10 than the values of τ reported by Wansleben and Landau [7].

If the effects of the central perturbation are transmitted by local means, that is a perturbed site can only affect its nearest neighbours, then z has a lower bound of unity. Hence $z = 1$ corresponds to the 'speed of light' for the transmission of information in the lattice. If damage can spread by non-local means then z may be less than unity [9], so that $z < 1$ corresponds to 'action at a distance'. If spin flips in the system are completely random and independent then $z = 2$, which indicates that the damage moves through the lattice like a diffusive, random walk process. If there is a net repulsion between the damaged sites and the unaffected neighbours then z may be less than 2, indicating that the damage spreading process is subject to an effect analogous to the excluded volume repulsion experienced by a self-avoiding walk. Correspondingly, if there is a net attraction created by the damaged sites, then z may be greater than 2, which is reminiscent of the anomalous diffusion of a random walker on a percolating cluster. Our numerical results suggest that z is near the random walk limit in $d = 3$, while there is a net attraction caused by the damaged sites in $d = 2$, where $z > 2$.

The damage concept is a very sensitive means of detecting fluctuations. It is used in the following manner. The system evolves at T_c to equilibrium and a replica is made of this equilibrium configuration. Then a central site σ_c is chosen and fixed permanently up in system A and permanently down in system B:

$$\sigma_c^A = -\sigma_c^B = 1. \tag{3}$$

The probability, $p(\uparrow)$ of finding a typical spin in the up ($\uparrow$) state is

$$p(\uparrow) = \frac{\exp(-E(\uparrow)/kT)}{\exp(-E(\uparrow)/kT) + \exp(-E(\downarrow)/kT)} \tag{4}$$

where $E(\uparrow)$ is the energy of the system when the site in question is up, and $E(\downarrow)$ is the energy when this site is down. The same random number is used when equivalent sites in both systems are checked to determine their new states. Note that spins at equivalent sites in both replicas will flip together unless the perturbation has influenced their neighbours. When this occurs there is a finite probability for the damage to propagate. The damaged sites, i.e. those sites whose spins are in different states in the replicas, allow for the clean detection of a fluctuation, seeded from the initial damage. We monitor the following two quantities: the average time taken for a fluctuation to touch the edge of the lattice for the first time, and the actual number of damaged sites found in the system at this time.

Statics. We measured the fractal dimensionality of the damage created by keeping the boundaries permanently damaged, i.e. the spins of the boundary sites of system A are kept permanently up while those of system B are kept permanently down. The average internal damage is the magnetisation [5] and at T_c this should scale with the fractal dimensionality $d_f = d - \beta/\nu$, with respect to L. We confirm the $d = 2$ results already reported in [5], i.e. $d_f = 1.88 \pm 0.02$, while in $d = 3$ we find $d_f = 2.49 \pm 0.05$. There are very strong finite size effects in $d = 3$ and we note that d_f is strongly dependent on the range of L used in its determination. We have obtained the value $d_f = 2.49 \pm 0.05$ from extrapolating a plot of $d_f(L)$ against $1/\sqrt{L}$ for the finite systems considered here, to infinite L. This plot shows less curvature than that of $d_f(L)$ against $1/L$.

Dynamics. We measured dynamic properties using damage initiated at a single central site in the system. The appropriate exponents, z, d_{act} and α are defined by the following relations:

$$\langle \tau_e \rangle \sim L^z \qquad (\xi = \infty \text{ at } T = T_c) \tag{5}$$

$$\langle s \rangle \sim L^{d_{act}} \tag{6}$$

and

$$\langle s \rangle \sim \langle \tau_e \rangle^{\alpha} = \langle \tau_e \rangle^{d_{act}/z} \tag{7}$$

where $\langle \tau_e \rangle$ is the average touching time, $\langle s \rangle$ the average damage at this time and L is the lattice size. Damage spreading has been used to determine the dynamic exponent, z for various models; the Kauffman model in two [2] and three [10] dimensions, the Ising model with unnormalised Glauber or Metropolis dynamics [4], and for $d = 2$ Ising systems with ferromagnetic and spin-glass interactions subjected to heat bath dynamics [3].

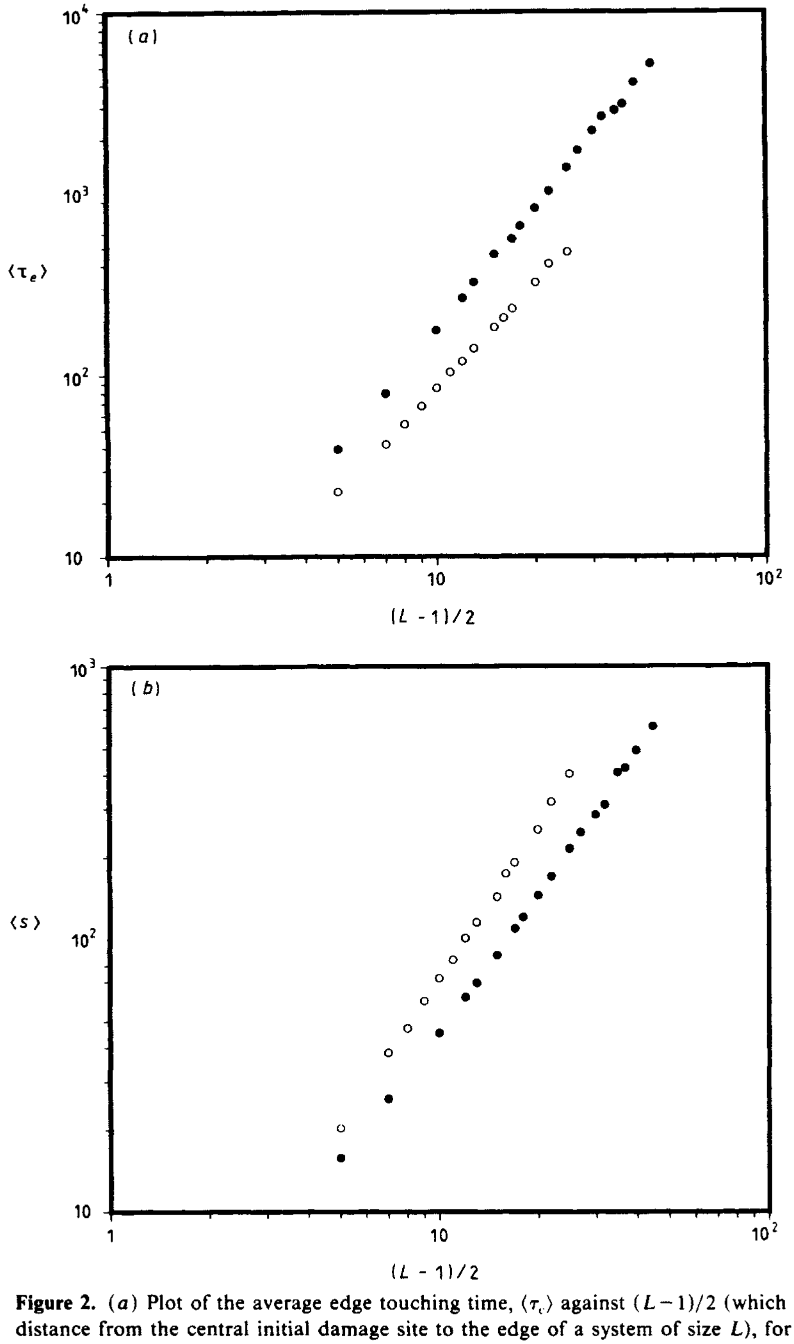

Figure 2. (*a*) Plot of the average edge touching time, $\langle \tau_e \rangle$ against $(L-1)/2$ (which is the distance from the central initial damage site to the edge of a system of size L), for Ising models with heat bath dynamics in $d=2$ (●) and $d=3$ (○). (*b*) Plot of the average number of damaged sites at the edge touching time, $\langle s \rangle$ against $(L-1)/2$ for $d=2$ (●) and $d=3$(○) Ising models.

Figure 2(a) shows the average touching time for the $d=2$ Ising model at T_c as a function of L ($10 \leqslant L \leqslant 100$). A least squares fit of the data leads to a value of $z=2.24 \pm 0.04$. The average number of damaged sites at the touching time is shown in figure 2(b) and the least squares fit of the data leads to a slope of $d_{act}=1.72 \pm 0.03$. The exponent α, which we believe is there reported for the first time for the Ising model is $\alpha=0.77 \pm 0.11$. In $d=3$ we have assumed a value [11] of $T_c=4.511\,6174$ and the equivalent data are shown in figure 2(a) and (b). A least squares fit of the data yields $z=2.02 \pm 0.03$, $d_{act}=1.85 \pm 0.05$ and $\alpha=0.92 \pm 0.10$.

In order to observe the finite size effects, we examine also the propagation of damage in relatively large systems ($L=101$ in $d=2$, and $L=51$ in $d=3$), but now focusing on the time taken for damage to reach certain internal distances, L', where $L' \leqslant (L-1)/2$. The results we obtain with this technque are in good agreement with those from the finite size analysis above although the error bars are somewhat larger. The data for the different internal sizes L' in a particular trial are correlated so that far more independent trials are needed to provide accurate exponents. A reassuring feature is that finite size effects due to boundary conditions appear to be negligible.

We have introduced a new method for the determination of the dynamic exponent z and have defined and measured an exponent α which describes the growth of a damage cloud with time. The $d=3$ result is in good agreement with the values obtained by the traditional Monte Carlo approach and renormalisation group techniques. The value $z=2.24$ in $d=2$ is somewhat higher than the commonly accepted value of $z=2.13$ but agrees with some of the earlier results found by Achiam [12] and by Miyashita and Takano [13]. The method is clean and precise in the determination of the touching time, but errors may occur due to an inadequate number of trials (500 trials for the large systems). In spite of this limitation the average quantities appear to be well converged and the value of z reported here may be considered with confidence. The exponent describing the growth of clusters is $\alpha=0.77$ ($d=2$) and $\alpha=0.92$ ($d=3$). Intuitively one expects that $d_{act} \simeq d_f$ but this is not the case since damaged sites may 'heal' and at the touching time there is rarely a spanning cluster. In fact, when measuring d_{act}, we have sampled from the ensemble of all clusters (i.e. $\langle s \rangle = \Sigma\, s^2 n_s$) and hence $d_{act}=\gamma/\nu$ and not d_f. If we consider a line of damage spanning the lattice in one direction and measure the equilibrium damage at T_c then the damage sites should scale with L as the fractal dimensionality, since we now have a spanning cluster by construction. It has also been indicated [14] that careful extrapolation of the mass of the actual damage at touching time does confirm that $d_{act} \simeq d_f$.

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